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Research Paper

On Convergence Criteria for Sequences

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ABSTRACT

In this paper we discuss the concept of convergence of real, complex and functions $\{f_n\}$ sequences, also we discuss the concept of sub-sequences. We presented the concept of convergence criteria for the sequences. First, we presented the cauchy criterion for convergence, and then we presented Weierstrass M-test for convergence and its some applications.

Key words: convergence – sequences - Weierstrass M-test-cauchy criterion

1. Sequences and Convergence

Definition 1.1. A sequence is a function [2,5] whose domain is $N$and whose codomainis $\mathbb{R}$. Given a function $f: N \to \mathbb{R}, f(n)$ is the nth term in the sequence.

Example1.2. Let$x_n = \frac{1}{n}$. In this case, our function$f$is defined as

$$f(n) = \frac{1}{n}$$

As a listed sequence of numbers, this would look like the following:

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\dots\right)$$

Definition1.3. A sequence of real numbers converges [1, 4] to a real number $a$if, forevery positive number ε, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$. We call such an $a$the limit of the sequence and write $\lim_{n\to\infty} a_n(x) = a$.

Definition1.4. A sequence $(f_n)_{n=1}^{\infty}$of functions [4] on a subset$\boldsymbol{A}$ of$\mathbb{R}$ into $\mathbb{R}$.

Definition1.5. (*Pointwise convergence*), [4] Let $\boldsymbol{D}$be a subset of $\mathbb{R}$and let $\{f_n\}$be a sequence of functionsdefined on D. We say that $\{f_n\}$converges pointwise on $\boldsymbol{D}$if$\lim_{n\to\infty} f_n(x)$ exists for each point $x$in $\boldsymbol{D}$.

This means that $\lim_{n\to\infty} f_n(x)$ is a real number that depends only on x.

If $\{f_n\}$is pointwise convergent then the function defined by

$f(x) = \lim_{n\to\infty} f_n(x)$for every $x$in $\boldsymbol{D}$, is called the pointwise limit of the sequence $\{f_n\}$

Note: The notation $N = N(x, \varepsilon)$ means that the natural number Ndependson the choice of $x$andε.

Definition1.6. (*Uniform convergence*), [4, 5] Let $\boldsymbol{D}$ be a subset of $\mathbb{R}$ and let $\{f_n\}$ be a sequence of realvalued functions defined on $\boldsymbol{D}$. Then $\{f_n\}$converges uniformly to f if givenany $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that$|f_n(x) - f(x)| < \varepsilon$ for every $n > N$ and for every x in $\boldsymbol{D}$.

Note: In the above definition the natural number N depends only on ε.

Therefore, uniform convergence implies pointwise convergence.

2. Subsequences

Definition 2.1.

Let $\{a_n\}_{n\geq 1}$be a sequence and$\{n_k\}_{k\geq 1}$ any strictly increasing sequence of positive integers; [2] that is,

$$0 < n_1 < n_2 < n_3 < \cdots.$$

Then the sequence $\{a_{n_k}\}_{k\geq 1}$, i.e., $\{b_k\}_{k\geq 1}$, where $b_k = a_{n_k}$, is called a subsequence of $\{a_n\}_{n\geq 1}$. That is, a subsequence is obtained by choosing terms from the original sequence, without altering the order of the terms, through the map $k \to n_k$, which determines the indices used to pick out the subsequence.For instance, $\{a_{7k+1}\}$ corresponds to the sequence of positive integers$n_k = 7k + 1, k = 1,2,\ldots$.

Observe that everyincreasing sequence $\{n_k\}$ofpositive integers must tend to infinity, because

$$n_k \geq k \; for \; k = 1,2,\ldots.$$

The sequences

$$\left\{\frac{1}{k^2}\right\}_{k\geq 1}, \left\{\frac{1}{2k}\right\}_{k\geq 1}, \left\{\frac{1}{2k+1}\right\}_{k\geq 1}, \left\{\frac{1}{5k+3}\right\}_{k\geq 1}, \left\{\frac{1}{2^k}\right\}_{k\geq 1}$$

are some subsequences [2] of the sequence $\{1/k\}_{k\geq 1}$, formed by setting $n_k = k^2, 2k, 2k+1, 5k+3, 2^k$, respectively.

Note that all the abovesubsequencesconverge to the same limit, which is also the limit of the original sequence$\{1/k\}_{k\geq 1}$. Can we conjecture that every subsequence of a convergent sequence must converge and converge to the same limit?

Theorem 2.2. (*Invariance property of subsequences*). [2]

If $\{a_n\}$ converges, then every subsequence $\{a_{n_k}\}$ of it converges to the same limit. Also,if $a_n \to \infty$, then $\{a_{n_k}\} \to \infty$ as well.

Proof. Suppose that $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. Note that $n_k \geq k$. Let$L = \lim a_n$ and $\varepsilon > 0$ be given. Then there exists an N such that

$$|a_k - L| < \varepsilon \; for \; k \geq N. (1)$$

Now $k \geq N$ implies $n_k \geq N$, which in turn implies that

$$|a_{n_k} - L| < \varepsilon \; for \; n_k \geq N. (2)$$

Thus, a_{n_k} converges to L as $k \to \infty$. The proof of the second partfollowssimilarly.

Corollary2.3. The sequence $\{a_n\}$ is divergent [4] if it has two conv-ergent subsequences with different limits. Also, $\{a_n\}$ is divergent if it has a subsequencethat tends to ∞ or a subsequence that tends to $-\infty$.

Theorem2.4. A sequence is convergent if and only if there exists a real number L such that every subsequence of the sequence has a further subsequencethat converges to L.

Corollary2.5. If both odd and even subsequences of $\{a_n\}$ converge to thesame limit l, then so does the original sequence.

Note that $\{(-1)^n\}$ diverges, because it has two subsequences $\{(-1)^{2n}\}$and $\{(-1)^{2n-1}\}$ converging to two different limits, namely 1 and -1.

3. Complex Sequences

Let $\{z_n\}$be a sequence of complex numbers [3] and let $z \in \mathbb{C}$. We say that $\{z_n\}$ converges to $z$and write $z_n \to z$ (or $\lim z_n = z$ etc.) if for every positive real number $\varepsilon > 0$, there exists anatural number N such that

$$n \geq N \Rightarrow |z_n - z| < \varepsilon$$

Theorem 3.1. Let $z_n = x_n + iy_n$.

(i) $z_n \to z \Rightarrow x_n \to \Re z, y_n \to \Im y$

(ii) $x_n \to x,\; y_n \to y \Rightarrow z_n \to x_n + iy_n$

Proof. (i) Put $x_n = \Re z$. $|x_n - X| = \Re(z_n - z) \leq |z_n - z|$. So given $\varepsilon > 0$ use the same N.

(ii)$|z_n - z| \leq |x_n - x| + |y_n - y|$ by Δ law

Find N_1 to ensure first term is less than $\varepsilon/2$, and N_2 to ensure second is less than$\varepsilon/2$then use $N := min(N_1, N_2)$.

4. Convergence criteria for sequences

I. Cauchy criterion

Definition4.1. [3] The real sequencea_n converges to something if and only if this holds: for every $\varepsilon > 0$ thereexists N such that $|a_n - a_m| < \varepsilon$whenever $n, m > N$.This is necessary and sufficient.

To prove one implication: Suppose the sequence a_n converges, [2] say to a. Then by definition, for every $\varepsilon > 0$we can find N such that

$|a - a_n| < \varepsilon$ whenever $n > N$. But then if we are given $\varepsilon > 0$ we can find N suchthat $|a - a_n| < \varepsilon/2$ for $n > N$, and then

$|a_n - a_m| = |(a_n - a) - (a_m - a)| < |a_n - a| + |a_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ (3)

for $m, n > N$.

To prove the other: Suppose the criterion [3] holds. We know that we have a subsequence a_{n_i} which convergesto some a. I claim that in fact the whole sequence converges to this same a. We know that for any $\varepsilon > 0$ we can find N_1 such that $|a_{n_i} - a| < \varepsilon$ for $i \geq N_1$. We also know that if we are given $\varepsilon > 0$ we can find K_2such that $|a_n - a_m| < \varepsilon$ for $, m \geq N_2$.

Now we want to prove that for any $\varepsilon > 0$ we can find N such that $|a_n - a| < \varepsilon$ for $n \geq N$.

First choose N_1 such that $|a - a_{n_i}| < \varepsilon/2$ for $i \geq N_1$. Second, choose N_2 such that

$|a_n - a_m| < \varepsilon/2$ (4)

for $m, n \geq N_2$. Suppose $n \geq N_2$. Choose some a_{n_i} with both $n_i \geq N_2$ and $i \geq N_1$. Then

$|a_n - a| = |(a_n - a_{n_i}) + (a_{n_i} - a)| \leq |a_n - a_{n_i}| + |a_{n_i} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ (5)

Now Suppose $\{z_n\}$is a sequence of complex numbers [3] for $n \in \mathbb{N}$. Then$\{z_n\}$ converges if and only if for any$\varepsilon > 0$ there exists $N \in \mathbb{Z}$an such that$|z_n - z_m| < \varepsilon$ for every $m, n \in \mathbb{Z}$such that $m > n > N$.

Any sequence that satisfies the Cauchy Criterion [3] is known as a Cauchy sequence. The above theorem also shows that every convergent sequence is Cauchy, and every Cauchy sequence is convergent.

Corollary 4.2.

If $\{z_n\}$is a Cauchy sequence [2, 3] that converges to z, and N is chosen such that $|z_n - z_m| < \varepsilon$for every$m, n \in \mathbb{Z}$ such that$m > N, n > N$, then for each$, n > N$ $,|z_n - z| < \varepsilon$. .

Proof:

This proof is rather straightforward. Let $m \to \infty$in the inequality $|z_n - z_m| < \varepsilon$. It follows from this that$|z_n - z| \leq \varepsilon$.

Corollary4.3.

The series$\sum_{k=0}^{\infty} a_k$ converges [2] if and only if for any$\varepsilon > 0$ there exists an N such that$|\sum_{k=n+1}^{m} a_k| < \varepsilon$ for every $m, n \in \mathbb{Z}$such that$m > n > N$

Definition 4.4. A sequence (f_n) of functions $f_n : \boldsymbol{A} \to \mathbb{R}$ is uniformly Cauchyon $\boldsymbol{A}$if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n > N$implies that $|f_m(x) - f_n(x)| < E$for all $x \in \boldsymbol{A}$.

The key part of the following proof is the argument to show that a pointwise convergent, uniformly Cauchy sequence converges uniformly.

Theorem 4.5. A sequence (f_n) of functions $f_n : \boldsymbol{A} \to \mathbb{R}$ convergesuniformly on$\boldsymbol{A}$if and only if it is uniformly Cauchy on $\boldsymbol{A}$.

Proof. Suppose that (f_n) converges uniformly, [4] to $f$on $\boldsymbol{A}$. Then, given $\mathcal{E} > 0$, thereexists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \mathcal{E}/2$ for all $x \in \boldsymbol{A}$if $n > N$.

It follows that if $m, n > N$ then

$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < E$

for all $x \in \boldsymbol{A}$,

which shows that (f_n) is uniformly Cauchy. [3,4]

Conversely, suppose that (f_n) is uniformly Cauchy. Then for each $x \in \boldsymbol{A}$, thereal sequence [2,5] $(f_n(x))$ is Cauchy, so it converges by the completeness of $\mathbb{R}$. Wedefine $f : \boldsymbol{A} \to \mathbb{R}$ by

$f(x) = lim_{n\to\infty} fn(x)$, (6)

and then $f_n \to f$pointwise.

To prove that$f_n \to f$uniformly, let $\mathcal{E} > 0$. Since (f_n) is uniformly Cauchy, wecan choose $N \in \mathbb{N}$ (depending only on $\mathcal{E}$) such that

$|fm(x) - fn(x)| < \mathcal{E}/2$for all $x \in \boldsymbol{A}$if $m, n > N$.

Let $n > N$and $x \in \boldsymbol{A}$. Then for every $m > N$we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \mathcal{E}/2 + |f_m(x) - f(x)|.$$

Since $f_m(x) \to f(x)$ as $m \to \infty$, we can choose $m > N$(depending on x, but itdoesn't matter since $m$doesn't appear in the final result) suchthat

$$|f_m(x) - f(x)| < \mathcal{E}/2$$

It follows that if $n > N$, then

$|f_n(x) - f(x)| < \varepsilon$ (7)

for all $x \in \boldsymbol{A}$,

which proves that $f_n \to f$uniformly.

Alternatively, we can take the limit as $m \to \infty$in the Cauchy condition to getfor all $x \in \boldsymbol{A}$and $n > N$that

$|f(x) - f_n(x)| = lim_{m\to\infty}|f_m(x) - f_n(x)| \leq \mathcal{E}/2 < E$ (8)

II. Weierstrass M-test

Theorem4.6. (*WeierstrassM-test*) [5]

Suppose $\{f_k\}$is a sequence of real- or complex-valued functions [3] on some set $\boldsymbol{E}$. Also, suppose that $\sum_{k=0}^{\infty} M_k$is a convergent series whereM_k are real non-negative terms. If $|f_k(z)| \leq M_k$for all k greater than some number N and for all z in some set $\boldsymbol{E}$, then it follows that the series$\sum_{k=0}^{\infty} f_k$ converges uniformly on $\boldsymbol{E}$.

Proof:

Since $\sum_{k=0}^{\infty} M_k$is Cauchy, we can choose a number$M > N$ such that for any m and n that satisfy $m > n > M$we get that$\sum_{k=n+1}^{m} M_k < \varepsilon$. Then we see that for z in the set $\boldsymbol{E}$ that our series$\sum_{k=0}^{\infty} f_k\ (z)$ is also Cauchy, since

$|\sum_{k=n+1}^{m} f_k(z)| \leq \sum_{k=n+1}^{m}|f_k(z)| \leq \sum_{k=n+1}^{m} M_k < \varepsilon$

(10)

Therefore,$\sum_{k=0}^{\infty} f_k\ (z)$ converges for every$z \in \boldsymbol{E}$. Let us say that $\sum_{k=0}^{\infty} f_k\ (z)$converges to the function$F(z)$. Now, we want to show that$\sum_{k=0}^{\infty} f_k\ (z)$ converges uniformly to$F(z)$. Observe that we can rewrite

$$\left|\sum_{k=n+1}^{m} f_k(z)\right| \leq \sum_{k=n+1}^{m} |f_k(z)| \leq \sum_{k=n+1}^{m} M_k < \varepsilon$$

in terms of partial sums

$|\sum_{k=0}^{m} f_k(z) - \sum_{k=0}^{n} f_k(z)| < \varepsilon$ (11)

for all $z \in \boldsymbol{E}$, and where$m > n > N$. Then applying Corollary (3.4) of the Cauchy Criterion, we see that

$|F(z) - \sum_{k=0}^{n} f_k(z)| \leq \varepsilon$ (12)

For$z \in \boldsymbol{E}$, and where $m > n > N$. Thus, the uniform convergence is shown.

Theorem4.7. (*Comparison test*) [2]

Suppose we have the terms $a_k$such that$|a_k| \leq M_k$ for all$k \in \mathbb{Z}$,$k > N$ for some number N. Then if the series $\sum_{k=0}^{\infty} M_k$converges, the series $\sum_{k=0}^{\infty} a_k$converges as well.

Since we know some of the ideas behind the Weierstrass M-Test, [5] we can now begin to look at some of its applications. We will first consider an *application* of the Weierstrass M-Test in the set of $\mathbb{R}$, before moving into applications within the set of$\mathbb{C}$.

Example4.8.

Show that the real-valued series

$$\sum_{k=1}^{\infty} \frac{1}{4^k} \sin\left(\frac{k}{3^k}\right)$$

is uniformly convergent.

The WeierstrassM-Test [5] gives us the ability to show this without considering any limits. First, we observe that for any$x \in \mathbb{R}$,$\left|\sin\left(\frac{k}{3^k}\right)\right| \leq 1$ for all k. Then it is easy to see that $\left|\frac{1}{4^k}\sin\left(\frac{k}{3^k}\right)\right| \leq \frac{1}{4^k}$. So now let $M_k = \frac{1}{4^k}$.

Now we want to show that the series$\sum_{k=1}^{\infty} \frac{1}{4^k}$ (our series $\sum_{k=1}^{\infty} M_k$) is convergent. This series $\sum_{k=0}^{\infty} \frac{1}{4^k}$converges to$\frac{1}{3}$ by the following Lemma.

Lemma 4.9.

The series $\sum_{k=0}^{\infty} a^k$converges to$\frac{1}{1-a}$ if $|a| < 1$.

So we now have$\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{4}{3} = 1 + \sum_{k=1}^{\infty} \frac{1}{4^k} = 1 + \frac{1}{3}$. Hence $\sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}$, (our series$\sum_{k=1}^{\infty} M_k = \frac{1}{3}$). Now by the Weierstrass M-Test we see the

series$\sum_{k=1}^{\infty}\frac{1}{4^k}\sin\left(\frac{k}{3^k}\right)$is uniformly convergent on$\mathbb{R}$.
Now to consider an *application* of the Weierstrass M-Test in the set of $\mathbb{C}$.

Example 4.10.
Show that the exponential function$f(z) = e^z$ is uniformly convergent [1,4] on any bounded set $\boldsymbol{S} \subset \mathbb{C}$.
Recall that e^zcan be rewritten as the series $\sum_{k=0}^{\infty}\frac{z^k}{k!}$. Now we will show that this series is uniformly on some disk $\boldsymbol{D}$ of radius r centered at the origin. To show this we must find someM_k such that $\left|\frac{z^k}{k!}\right| \le M_k$for all$z \in \boldsymbol{D}$. Recall that $|z^k| \le |z|^k$, and that $|z| \in \mathbb{R}$. So let$|z| < r \in \mathbb{R}$. Then it follows that $\left|\frac{z^k}{k!}\right| \le \frac{|z|^k}{k!} \le \frac{r^k}{k!}$. We see that $\frac{r^k}{k!} \in \mathbb{R}$, so now let $M_k = \frac{r^k}{k!}$.
We may be able to *apply* the Weierstrass M-Test, [5] if we can show that the series $\sum_{k=0}^{\infty} M_k$converges. If we use the (*Ratio Test*), [2] we can prove that $\sum_{k=0}^{\infty} M_k$is convergent. So now recall:

III. Ratio Test:
[2] Given a series $\sum_{k=0}^{\infty} a_k$, find

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L \ (13)$$

If $L > 1$, the series diverges
If $L < 1$, the series converges
If $L = 1$or the limit fails to exist, then the test is inconclusive.

So now we see that

$$\lim_{k\to\infty}\left|\frac{M_{k+1}}{M_k}\right| = \lim_{k\to\infty}\frac{\frac{r^{k+1}}{(k+1)!}}{\frac{r^k}{k!}} = \lim_{k\to\infty}\frac{r}{r+1} = 0.$$

Thus by the (Ratio Test)we see that the series $\sum_{k=0}^{\infty} M_k$converges. Then by the Weierstrass M-Test we see that$\sum_{k=0}^{\infty}\frac{z^k}{k!}$Is uniformly convergent on some disk $\boldsymbol{D}$ of radius $\boldsymbol{D}$ centered at the origin.

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