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Original Research Article

Analytic Differential Geometry with Manifolds

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ABSTRACT

In This paper develop the Analytic geometry of classical gauge theories, on compact dimensional manifolds, some important properties of fields k , the manifold structure $M \in C^{\infty}$ of the configuration space, we study the problem of differentially projection mapping parameterization system by constructing C^{∞} rank n on surfaces $n-1$ dimensional is sub manifold space R^{n-1}.

Index Terms: basic notion on differential geometry - differential between surfaces $M, N \subset R$ is called the differential manifolds- tangent and cotangent space- differentiable injective manifold- Operator geometric on Riemannian manifolds.

INTRODUCTION

The object of this paper is to familiarize the reader with the basic analytic of and some fundamental theorem in deferrable Geometry. To avoid referring to previous knowledge of differentiable manifolds, we include surfaces, which contains those concepts and result on differentiable manifolds which are used in an essential way in the rest of the. The first section II present the basic concepts of analytic Geometry (Riemannian metrics, Riemannian connections, geodesics and curvature). consists of understanding the relationship between geodesics and curvature, Jacobi fields an essential tool for this understanding, are introduced in we introduce the second fundamental from associated with an isometric immersion and prove a generalization of the theorem of Riemannian Geometry this allows us to real the notion of curvature in Riemannian manifolds to the classical concept of Gaussian curvature for surfaces. way to construct manifolds, a topological manifolds C^{∞} analytic manifolds, stating with topological manifolds, which are Hausdorff second countable is locally Euclidean space We introduce the concept of maximal C^{∞} atlas, which makes a topological manifold into a smooth manifold, a topological manifold is a Hausdorff, second countable is local Euclidean of dimension n. If every point p in M has a neighborhood U such that there is a homeomorphism φ from U onto a open subset of R^n. We call the pair a coordinate map or coordinate system on U. We said chart (U,φ) is centered at $p \in U$, $\varphi(p) = 0$, and we define the smooth maps $f: M \to N$ where M, N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M and (V_β, g_β) on N.

NOTIONS ON DIFFERENTIAL GEOMETRY

2.1 Basic analytic geometry

Definition 2.1.1

A topological manifold M of dimension n, is a topological space with the following properties:

(i) M is a Hausdorff space . For ever pair of points $p, g \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $g \in V$.
(ii) M is second countable . There exists accountable basis for the topology of M .
(iii) M is locally Euclidean of dimension n . Every point of M has a neighborhood that ishomeomorphic to an open subset of R^n .

Definition 2.1.2

A coordinate chart or just a chart on a topological $n-$manifold M is a pair (U, φ) , Where U is an open subset of M and $\varphi : U \to \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi\,(U) \subset R^n$.

Examples 2.1.3

Let S^n denote the (unit) $n-$sphere, which is the set of unit vectors in R^{n+1}: $S^n = \{x \in R^{n+1} : |\, x\, | = 1\}$ with the subspace topology, S^n is a topological $n-$manifold.

Definition 2.1.4

The $n-$dimensional real (complex) projective space, denoted by $P_n(R)$ *or* $P_n(C))$, is defined as the set of 1-dimensional linear subspace of R^{n+1} *or* $C^{n+1})$, $P_n(R)$ *or* $P_n(C)$ is a topological manifold.

Definition 2.1.5

For any positive integer n , the $n-$torus is the product space $T^n = (S^1 \times ... \times S^1)$ .It is an $n-$dimensional topological manifold. (The 2-torus is usually called simply the torus).

Definition2.1.6

The boundary of a line segment is the two end points; the boundary of a disc is a circle. In general the boundary of an $n-$manifold is a manifold of dimension $(n-1)$, we denote the boundary of a manifold M as ∂M . The boundary of boundary is always empty, $\partial\partial M = \phi$.

Lemma 2.1.7

(i) Every topological manifold has a countable basis of Compact coordinate balls.
(ii) Every topological manifold is locally compact.

Definitions 2.1.8

Let M be a topological space n-manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \phi$, the composite map

(1) $$\psi \circ \varphi^{-1} : \varphi\,(U \cap V) \to \psi\ \ (U \cap V)$$

Is called the transition map from φ to ψ .

Definition 2.1.9

A smooth structure on a topological manifold M is maximal smooth atlas. (Smooth structures are also called differentiable structure or C^∞ structure by some authors).

Definition 2.1.10

A smooth manifold is a pair $(M,$ A), where M is a topological manifold and A is smooth structure on M . When the smooth structure is understood, we omit mention of it and just say M is a smooth manifold.

Definition 2.1.11

Let M be a topological manifold.
(i) Every smooth atlases for M is contained in a unique maximal smooth atlas.
(ii) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is smooth atlas.

Definition 2.1.12

Let M be a smooth manifold and let p be a point of M . A linear map $X : C^\infty(M) \to R$ is called a derivation at p if it satisfies:

(2) $$X\,(\,fg\,) = [(\,f\,(\,p\,)Xg\,) + (\,g(\,p\,)Xf\,)]$$

Forall $f, g \in C^\infty(M)$. The set of all derivation of $C^\infty(M)$ at p is vector space called the tangent space to M at p , and is denoted by [T_pM]. An element of T_pM is called a tangent vector at p .

Lemma 2.1.13

Let M be a smooth manifold, and suppose $p \in M$ and $X \in T_pM$ If f is a cons and function, then $Xf = 0$. If $f(p) = g(p) = 0$, then $X(\,fp) = 0$.

Definition2.1.14

If γ is a smooth curve (a continuous map $\gamma : J \to M$, where $J \subset R$ is an interval) in a smooth manifold M , we define the tangent vector to γ at $t_\circ \in J$ to be the vector $\gamma'(t_\circ) = \gamma_*\left(\frac{d}{dt}|_{t_\circ}\right) \in T_{\gamma(t_\circ)}M$, where $\left[d/dt\right]|_{t_\circ}$ is the standard coordinate basis for $T_{t_\circ}R$. Other common notations for the tangent vector to γ are $\left[\gamma^*(t_\circ), \frac{d\gamma}{dt}(t_\circ)\right]$ and $\left[\frac{d\gamma}{dt}|_{t=t_\circ}\right]$. This tangent vector acts on functions by:

(3) $\gamma'(t_\circ)\, f = \left(\gamma_* \frac{d}{dt}|_{t_\circ}\right) f = \frac{d}{dt}|_{t_\circ}(f \circ \gamma) = \frac{d(f\circ\gamma)}{dt}(t_\circ)$.

Definition 2.1.15

Let V and W be smooth vector fields on a smooth manifold M. Given a smooth function $f : M \to R$, we can apply V to f and obtain another smooth function Vf, and we can apply W to this function, and obtain yet another smooth function $(WV)f = W(Vf)$. The operation $f \to WV\,f$, however, does not in general satisfy the product rule and thus cannot be a vector field, as the following for example shows let $V = \left(\frac{\partial}{\partial x}\right)$ and $W = \left(\frac{\partial}{\partial y}\right)$ on R^n, and let $f(x,y) = x, g(x,y) = y$. Then direct computation shows that $VW(f\ g) = 1$, while $(f\,V\,W\ g + g\ V\ W\ f) = 0$, so VW is not a derivation of $C^\infty(R^2)$. We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function $WV\,f$. Applying both of this operators to f and subtraction, we obtain an operator $[V,W]: C^\infty(M) \to C^\infty(M)$, called the Lie bracket of V and W, defined by $[V,W]\ f\ =\ (VW)f - (WV)f$. This operation is a vector field. The Smooth of vector Field is Lie bracket of any pair of smooth vector fields is a smooth vector field.

Lemma 2.1.16

The Lie bracket satisfies the following identities for all $V,W,X \in (M)$. Linearity:

$$\forall a,b \in R, \begin{cases} [\ aV + bW, X\ ] = a\ [V,X] + b\,[W,X] \\ [\ X, aV + bW\ ] = a\ [X,V] + b\,[X,W]. \end{cases}$$

(i) Ant symmetry $[V,W] = -\ [W,V]$.

(ii) Jacobi identity $[V,[\ W,X\ ]] + [\ W,[\ X,V]\ ] + [\ X,[V,W]\ ] = 0$

For $f,g \in C^\infty(M)$ $[fV, g\,W] = fg[V,W] + (fVg)W - (gWf)V$

2.3 Convector Fields

Let V be a finite – dimensional vector space over R and let V^* denote its dual space. Then V^* is the space whose elements are linear functions from V to R, we shall call them Convectors. If $\sigma \in V^*$ then $\sigma : V \to R$ for the any $v \in V$, we denote the value of σ on v by $\sigma(v)$ or by $\langle v, \sigma \rangle$. Addition and multiplication by scalar in V^* are defined by the equations:

$\{(\sigma_1 + \sigma_2)\ (v) = \sigma_1(v) + \sigma_2(v),\ (\alpha\sigma)\,(v) = \alpha\ (\sigma\ (v)\ )\}$

Where $v \in V$, $\sigma, \alpha\sigma \in V^*$ and $\alpha \in R$.

Proposition2.3.1

Let V be a finite- dimensional vector space. If $(E_1,..., E_n)$ is any basis for V, then the convectors $(\omega^1,...,\omega^n)$ defined by.

(5) $\omega^i(E_j) = \delta^i_j = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$,

Form a basis for V^*, called the dual basis to (E_j). Therefore, $\dim V^* = \dim V$.

Definition 2.3.2

AC^r – Convector field σ on M, $r \geq 0$, is a function which assigns to each $\beta \in M$ a convector $\sigma_p \in T^*_P(M)$ in such a manner that for any coordinate neighborhood U,ϕ with coordinate frames $E_1,.., E_n$, the functions $\sigma(E_i),\ i = 1,....., n,$ are of class C^r on U. For convenience, "Convector field" will mean C^∞ – convector field.

2.4 The Exponential Map Normal Coordinates

We have already seen that there are many differences between the classical Euclidean geometry and the general Riemannian geometry in the large. In particular we have seen examples in which one of basic axioms of Euclidean geometry no longer holds. Two distinct geodesic (real lines) may intersect in more than one point. The global topology of the manifold is responsible for this “failure”. In this we will define using the metric some special collections to being Euclidean. Let (M, g) be Riemannian manifold and U, an open coordinate neighborhood with coordinate $(x^1,..., x^n)$.We will try to find a local change in coordinate $(x^i \to y^i)$ in which the expression of the metric is as close are to the Euclidean metric $g_0 = i, j\,dy^i\,dy^j$. Let $q \in u$, be the point with coordinate $(0,...,0)$ via a linear we may as well assume that $g_{ij}(q) = i, j$. We would like “spread” the above equality to an entire neighborhood of q. To achieve this we try to find local coordinates y^J near q such that in these new coordinates the metric is Euclidean up to order one i,e .

(6) $$\begin{cases} g_{i,j}(q) = \dfrac{\partial g_{ij}}{\partial y^k}(q) \\ = \dfrac{\partial_{i,J}}{\partial y^k} = \dfrac{\partial_{i,j}}{\partial y^k}(q) = 0 \ , \forall : i,j,k \in g \end{cases}$$

We now describe a geometric way of producing such coordinates using the geodesic flow .Denote as usual the geodesic from q with initial direction $X \in T_q(M)$. By $X_q(t)$ Not the following simple fact L $X \in V$. Hence, there exists a small neighborhood V of $T_q(M)$, Such that, for any $X \in V$, the geodesic $X_q(t)$ is defined for all $|t| \le 1$ .we define the exponential map at q.

$\exp_q : V \subset T_q(M) \to M \ , X \to X_q(1)$

The tangent space $T_q(M)$ is a Euclidean space, and we can define $D_q(r) \subset T_q(M)$, the open "disk" of radius r centered at the origin we have the following result centered at the origin .we have the following result

In particular, $dx^i(X_a) = h^i$, that is, dx^i measures the change in theis coordinate of a point as it moves from the initial to the terminal point of X_a. The preceding formula may thus be written.

(7) $$df(X_a) = \left(\frac{\partial f}{\partial x^1}\right)_a dx^1(X_a) + \dots + \left(\frac{\partial f}{\partial x^n}\right)_a dx^n(X_a).$$

This gives us a very good definition of the differential a function on $U \subset R^n$; is a field of linear functions which at any point a of the domain of f assigns to each vector X_a a number. Interpreting X_a as the displacement of the n independent variables from a, that is, a as initial point and $a+h$ as terminal point. $df(X_a)$ a approximates (linearly) the change in f between these points.

Definition 2.4.2

A convector tensor on a vector space V is simply a real valued $\phi(v_I, \dots, v_r)$ of several vector variables $v_I, \dots, v_r$ of V, linear in each separately.(i.e. multiline). The number of variables is called the order of the tensor. A tensor field ϕ of order r on a manifold M is an assignment to each point $P \in M$ of a tensor ϕ_P on the vector space $T_P(M)$, which satisfies a suitable regularity condition $C^0, C^r, or C^\infty$ as P varies on M.

Theorem 2.4.3

With the natural definitions of addition and multiplication by elements of R the set $(V)^r_s$ of all tensors of order (r,s) on V forms a vector space of dimension n^{r+s}.

Theorem 2.4.4

The maps A and S are defined on $(M)^r$ a C^∞- manifold and $(M)^r$ the C^∞- covariant tensor fields of order r , and they satisfy properties there. In these cases of (c), $F^* : {}^r(N) \to {}^r(M)$ is the linear map induced by a C^∞ mapping $F : M \to N$.

Definition 2.4.5

Let V be a vector space and $\varphi \in V$ are tensors. The product of φ and ψ, denoted $\varphi \otimes \psi$ is a tensor of order $r+s$ defined by : $\varphi \otimes \psi(v_1, \dots v_r, \dots v_{r+1}, \dots v_{r+s}) = \varphi(v_1, \dots, v_r)\psi(v_{r+1}, \dots, v_{r+s})$.

The right hand side is the product of the values of φ and ψ .The product defines a mapping $(\varphi, \psi) \to \varphi \otimes \psi$ of x ${}^r(V) \to {}^{r+s}(V)$.

Theorem 2.4.6

The product ${}^r(V)$ o ${}^r(V) \to {}^{r+s}(V)$ just defined is bilinear and associative. If $(\omega^1, \dots, \omega^n)$ is a basis of.

Definition 2.4.7

Carton's wedge product, also known as the exterior Product, as the ant symmetric tensor product of cotangent space basis elements $dx \wedge dy = 1/2 \ (dx \otimes dy - dy \otimes dx) = -dy \wedge dx$.

Note that, by definition, $dx \wedge dx = 0$. The differential line elements dx and dy are called differential 1-forms or 1-form; thus the wedge product is a rule for construction g 2-forms out of pairs of 1-forms.

Remark 2.4.8

Let α_p be an element of Λ^p α_p, β_p an element of Λ^q. Then $(\alpha_p \wedge \beta_q) = (-1)^{pq}(\beta_q \wedge \alpha_p)$. Hence odd forms ant commute and the wedge product of identical 1-forms will always vanish.

Definition 2.4.9

A topological space M is called (Hausdorff) if for all $x, y \in M$ there exist open sets such that $x \in U$ and $y \in V$ and $U \cap V = \phi$

Definition 2.4.10

A topological space M is second countable if there exists a countable basis for the topology on M .

Definition 2.4.11

A topological space M is locally Euclidean of dimension n if for every point $x \in M$ there exists on open set $U \in M$ and open set $w \subset R^n$ so that U and W is (homeomorphism).

Definition 2.4.12

A topological manifold of dimension n is a topological space that is Hausdorff, second countable and locally Euclidean of dimension N.

Definition 2.4.13

A smooth atlas A of a topological space M is given by:

(i) An open covering $\{U\}_{i \in I}$ where $U_i \subset M$ Open and $M = (\cup_{i \in I} U_i)$

(ii) A family $\{\phi_i : U_i \to W_i\}_{i \in I}$ of homeomorphism ϕ_i onto opens subsets $W_i \subset R^n$ so that if $U_i \cap U_j \neq \phi$ then the map $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is (Adiffoemorphism)

Example 2.4.14

The stereographic is map π on $\pi : S^2 \to \{N\}$ onto R^2 the noth pole $(0,0,1)$ $p \in S^2 \to \{N\}, \pi(p)$ is defined to be the point at which line N and p intersects the xy-plan $\pi : S^2 \longrightarrow \{N\} \longrightarrow R$ is "diffeomorphism" to do so write explicitly in coordinates and solute for π^{-1}

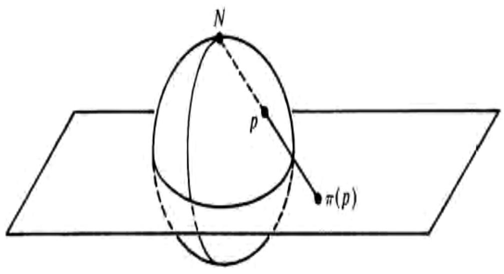


Fig. (1): the diffeomorphism

Definition 2.4.15

A smooth structure on a Hausdorff topological space is an equivalence class of atlases, with two atlases A and B being equivalent if for $(U_i, \varphi_i) \in A$ and $(V_j, \Psi_j) \in B$ with $U_i \cap V_j \neq \varphi$ then the transition map $\phi_i(U_i \cap V_j) \to \Psi_j(U_i \cap V_j)$ is a diffeomorphism (as a map between open sets of R^n).

Definition 2.4.16

A smooth manifold M of dimension n is a topological manifold of dimension n together with a smooth structure.

Definition 2.4.18

A map $F : M \to N$ is called a diffeomorphism if it is smooth objective and inverse $F^{-1} : N \to M$ is also smooth.

Definition 2.4.19

A map F is called an embedding if F is an immersion and homeomorphism onto its image

Definition 2.4.20

If $F : M \to N$ is an embedding then $F(M)$ is an immersed sub manifoldsof N .

Example 2.1.21

The vector function as vector fields on $R \subset E$, the function $f_i(t)$ is vector fields $r = tu + a$, $t \in R$ is parameterizations $u = u_i$ on line $a = a_i$ is vector as point on line L .

$$r = r(t) = (tu_1 + a_1, tu_2 + a_2, tu_3 + a_3)$$

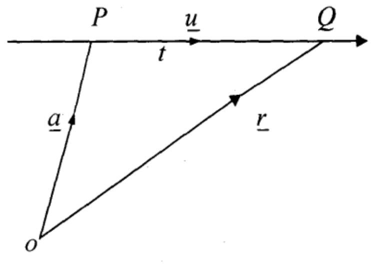


Fig.(2) : vector fields

2.5: Tangent space and vector fields

Let $C^\infty(M, N)$ be smooth maps from M and N and let $C^\infty(M)$ smooth functions on M is given a point $p \in M$ denote, $C^\infty(p)$ is functions defined on some open neighborhood of p and smooth at p .

Definition 2.5.1

(i) The tangent vector X to the curve $c:(-\varepsilon,\varepsilon)\to M$ at $t=0$ is the map $c(0):C^{\infty}(c(0))\to R$ given by the formula.

(8) $\left\{X(f)=c(0)(f)=\left(\frac{d(f\circ c)}{dt}\right)_{t=0} \quad \forall f\in C^{\infty}c(0)\right\}$

(ii) A tangent vector X at $p\in M$ is the tangent vector at $t=0$ of some curve $\alpha:(-\varepsilon,\varepsilon)\to M$ with $\alpha(0)=p$ this is $X=\alpha'(0):C^{\infty}(p)\to R$.

Remark 2.5.2

A tangent vector at p is known as a liner function defined on $C^{\infty}(p)$ which satisfies the (Leibniz property)

(9) $\begin{cases} X(f\,g)=X(f)g+f\,X(g) \\ \forall\, f,g\in C^{\infty}(p)\end{cases}$

Differential 2.5.3

Given $F\in C^{\infty}(M,N)$ and $p\in M$ and $X\in T_pM$ choose a curve $\alpha:(-\varepsilon,\varepsilon)\to M$ with $\alpha(0)=p$ and $\alpha'(0)=X$ this is possible due to the theorem about existence of solutions of liner first order ODEs , then consider the map $F_{*p}:T_PM\longrightarrow T_{F(P)}N$ mapping $X\to F_{*p}(X)=(F\circ\alpha)'(0)$, this is liner map between two vector spaces and it is independent of the choice of α.

Definition 2.5.5

The liner map F_{*p} defined above is called the derivative or differential of F at p while the image $F_{*p}(X)$ is called the push forward X at $p\in M$ **Definition 2.5.6**

Given a smooth manifold M a vector field V is a map $V:M\to TM$ mapping $p\to V(p)\equiv V_p$ and V is called smooth if it is smooth as a map from M to TM.

$X(M)$ Isan R vector space for $Y,Z\in X(M)$, $p\in M$ and $a,b\in R$,$(aY+bZ)_p=aV_p+bZ_p$ and for $f\in C^{\infty}(M)$,$Y\in X(M)$ define $fY:M\to TM$ mapping.

(10) $\{\,p\longrightarrow(fY)_p=f(p)Y_p\,\}$

2.6: Cotangent smooth n-manifolds

Let M be a smooth n-manifolds and $p\in M$ .We define cotangent space at p denoted by T_p^*M to be the dual space of the tangent space at $p:T_p^*(M)=\{f:T_pM\to R\}$, f smooth Element of T_p^*M are called cotangent vectors or tangent convectors at p.

(i) For $f:M\to R$ smooth the composition $T_p^*M\to T_{f(p)}R\cong R$ is called $(df)_p$ and referred to the differential of f .Not that $df_p\in T_p^*M$ so it is a cotangent vector at p.

(ii) For a chart (U,ϕ,x^i) of M and $p\in U$ then $\{dx^i\}_{i=1}^n$ is a basis of T_p^*M in fact $\{dx^i\}$ is the dual basis of $\left\{\frac{d}{dx^i}\right\}_{i=1}^n$.

Definition 2.6.1

The elements in the tensor product $V_s^r=(V\otimes....\otimes V\otimes V^*\otimes...\otimes V^*)$ are called tensors or r-contra variant, s- contra varianttensor.

Remark 2.6.2

The Tensor product is bilinear and associative however it is in general not commutative that is $(T_1\otimes T_2)\neq(T_2\otimes T_1)$ in general.

Definition 2.6.3

$T\in V_s^r$ is called reducible if it can be written in the form $T=V_1\otimes...\otimes V_r\otimes L^1\otimes...\otimes L^s$ for. $V_i\otimes V_r, L^j\in V^*$ for $1\le i\le r$,$1\le j\le s$.

Definition 2.6.4

Choose two indices (i,j) where $1\le i\le r$,$1\le j\le s$ for any reducible tensor $T=(V_1\otimes...\otimes V_r)\otimes(L^1\otimes....\otimes L^2)$ let $C_i^r(T)\in V_{s-1}^{r-1}$ We extend this linearly to get a linear map $C_i^j:(V_s^r\to V_{s-1}^{r-1})$ which is called tensor-contraction.

Definition 2.6.5

Let $F:M\to N$ be a smooth map between two smooth manifolds and $w\in\Gamma(T_k^0N)$ be a k covariant tensor field we define a k covariant tensor field F^*w over M by.

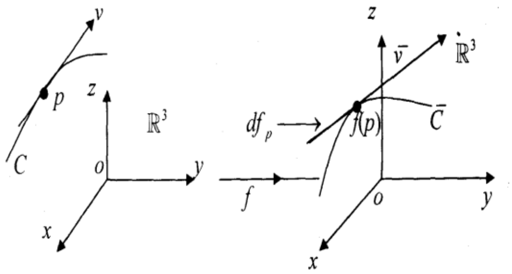
(11) $\begin{pmatrix}(F^*w)_p(v_1,...,v_k)=w_{F(p)}(F_{*p}(v_1),...,F_{*p}(v_k)) \\ \forall v_1,...,v_k\in T_pM\end{pmatrix}$

In this case F^*w is called the pullback of w by F.

Example 2.6.7

The tangent bundle section is function $f:R^n\to R^m$ is differential or tangent map as point p on tangent felids $df(p)$ is image df_p .

$df_p(v) = v_{f(p)}$, $C = f(C), p \in C \to f(p) \in \overline{C}$

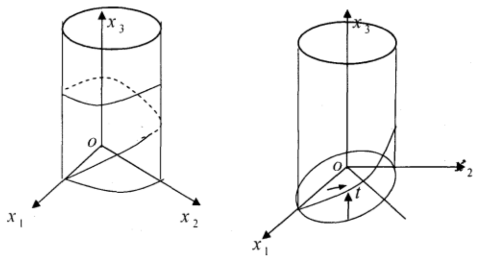


2.7: Integration of differential forms

$\int_M w$ is well defined only if M is orient able $\dim(M) = n$ and has a partition of unity and w has compact support and is a differential n-form on M .

Example 2.7.1

The circular helix on curve is parameters on



Definition 2.7.2

A pair (M, g) of a manifold M equipped with a Riemannian metric g is called a Riemannian manifold.

Definition 2.7.3

Suppose (M, g) is a Riemannian manifold and $p \in M$ we define the length (or norm) of a tangent vector $v \in T_pM$ to be $|v| = \sqrt{\langle v, v \rangle_p}$ Recall $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and the angle v, w between $v, w \in T_pM$ $(v \neq 0 \neq w)$ by

(12) $$\left(\cos(v, w) = \frac{\langle v, w \rangle_p}{|v||w|} \right).$$

Examples 2.7.4

(i).Euclidean metric (canonical metric) g_{Eucl} on R^n .

(13) $$\begin{Bmatrix} g_{Eucl} = \delta_{ij} dx^i \otimes dx^j = dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n \\ = dx^1 dx^1 + \ldots + dx^n dx^n \end{Bmatrix}$$

(ii) Induced metric

Let (M, g) be a Riemannian manifold and $f : N \to (M, g)$ an immersion where N is a smooth manifold (that is f is a smooth map and f is injective) then induced metric on N is defined .

(14) $$\begin{cases} (f\,g)_p(v, w) = g_{f(p)}[(f_*(v)), (f_*(w))] \\ : \qquad \forall v, w \in T_pN, p \in N \end{cases}$$

The induced metric S^n sometimes denoted $(g_{Eucl})|_{S^n}$ from the Euclidean space R^{n+1} and g_{Eucl} by the inclusion $i : S^2 \to R^{n+1}$ is called the standard (or round) metric on S^n clearly i is an immersion .Consider stereographic projection $S^2 \to R^3$ and denote the inverse map $u : R^2 \to S^2$ then $u^* g_{Eucl}$. Given the Riemannian metric for R^2 .

(iii) Product metricIf (M_1, g_1), (M_2, g_2) are two Riemannian manifolds then the product $M_1 \times M_2$ admits a Riemannian metric $g = g_1 \oplus g_2$ is called the product metric defined by .

(15) $$g(u_1 \oplus u_2, v_1 \oplus v_2) = g_1(u_1, v_1) \oplus g_2(u_2, v_2)$$

$$g(u_1 \oplus u_2, v_1 \oplus v_2) = g_1(u_1, v_1) \oplus g_2(u_2, v_2) .$$

Where $u_i, v_i \in T_{pi}M_i$ for $i = 1,2,....$ we use the fact that $T_{p_1, p_2}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_1}M_2$.

(iv)Warped product Suppose (M_1, g_1), (M_2, g_2) are two Riemannian manifolds then $(M_1 \times M_2, g_1 \oplus f^2 g_2)$ is the warped product of g_1, g_2 or denoted $(M_1, g_1) \times_f (M_2, g_2)$ where $f : M_1 \to R$ a smooth positive function is.

(16) $$\begin{Bmatrix} (g_1 \oplus f^2 g_2)_{p_1 p_2}(u_2 \oplus u_2, v_1 \oplus v_2) \\ = g_{1_{p_1}}(u_1, v_1) \oplus f(p_1) g_{2_{p_1}}(v_2, w_2) \end{Bmatrix}$$

Definition 2.7.6

A smooth map $f : (M, g) \to (N, h)$ between two Riemannian manifolds is called a conformal map with conformal factor $\lambda : (M \to R^+)$ if $(f^* h) = \lambda^2 g$. A conformal map preserves angles that is $(v, w) = (f_*(v), f_*(w))$ for all $u, v \in T_pM$ and $p \in M$.

Example 2.7.7

$S^2 \subset R^3$ We consider stereographic projection $S^2 / p_n \to R^2$. As stereographic projection is a diffeomorphism its inverse $u : R \to S / p_n$ is a conformal map. It follows from an exercise sheet that u is a conformal map with conformal factor $\rho(x, y) = 2/(1 + x^2 + y^2)$.

Definition 2.7.8

A Riemannian manifold (M, g) is locally flat if for every point $p \in M$ there exist a conformal diffeomorphism $f : U \to V$ between an open neighborhood U of p and $V \subset R^n$ of $f(p)$.

Definition 2.7.9

Given two Riemannian manifold (M, g) and (N, h) they are called isometric of there is a diffeomorphism $f : M \to N$ such that $f^* h = g$ such that a differ-morphism f is called an isometric.

Remark 2.7.10

In particular an isometrics $f :(M, g) \to (M, g)$ is called an isometric of (M, g). All isometrics on a Riemannian manifold from a group.

Definition 2.7.11

$(M, g), (N, h)$ Are called locally isometric if for every point $p \in M$ there is an isometric $f : U \to V$ from an open neighborhood U of p in M and an open neighborhood V of $f(p)$ in N.

Definition 2.7.12

Suppose $f :(M, g) \to (N, h)$ is an immersion then f is isometric if $f^* h = g$.

Definition 2.7.13

A bundle metric h on the vector bundle (E, M, π) is an element of $\Gamma(E^* \otimes E^*)$ which is symmetric and positive definite.

2.8: Differentiable injective manifold

The basically an m-dimensional topological manifold is a topological space M which is locally homeomorphism to R^m definition is a topological space M is called an m-dimensional (topological manifold) if the following conditions hold.

(i) M is a hausdorff space.

(ii) For any $p \in M$ there exists a neighborhood U of P which is homeomorphism to an open subset $V \subset R^m$.

(iii) M has a countable basis of open sets, coordinate charts (U, φ)

(iv) is equivalent to saying that $p \in M$ has a open neighborhood $U \in P$ homeomorphism to open disc D^m in R^m, axiom (v) says that M can covered by countable many of such neighborhoods , the coordinate chart (U, φ) where U are coordinate neighborhoods or charts and φ are coordinate .

A homeomorphisms , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$, which are again homeomorphisms by definition , we usually write $p = \varphi^{-1}(x), \varphi : U \to V \subset R^n$ as coordinates for U and $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ as coordinates for U , the coordinate charts (U, φ) are coordinate neighborhoods, or charts , and φ are coordinate homeomorphisms, transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$ which are again homeomorphisms by definition , we usually $x = \varphi(p), \varphi : U \to V \subset R^n$ as a parameterization U. A collection $A = \{(\varphi_i, U_i\}_{i \in I}$ of coordinate chart with $M = \cup_i U_i$ is called atlas for M. The transition maps φ_{ij} a topological space M is called (hausdorff) if for any pair $p, q \in M$, there exist open neighborhoods $p \in U$ and $q \in U'$ such that $U \cap U' \neq \phi$ for a topological space M with topology $\tau \in U$ can be written as union of sets in β , a basis is called a countable basis β is a countable set .

Definition 2.8.1

Let X be a set a topology U for X is collection of X satisfying.

(i) ϕ And X are in U

(ii) The intersection of two members of U is in U .

(iii) The union of any number of members U is in U . The set X with U is called a topological space the members $U \in u$ are called the open sets. let X be a topological space a subset $N \subseteq X$ with $x \in N$ is called a neighborhood of x if there is an open set U with $x \in U \subseteq N$, for example if X a metric space then the closed ball $D_\varepsilon(x)$ and the open ball $D_\varepsilon(x)$ are neighborhoods of x a subset C is said to closed if $X \setminus C$ is open

Definition 2.8.2

A function $f : X \to Y$ between two topological spaces is said to be continuous if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X .

Definition 2.8.3

Let X and Y be topological spaces we say that X and Y are homeomorphism if there exist continuous function such that $f \circ g = id_y$ and $g \circ f = id_X$ we write $X \cong Y$ and say that f and g are homeomorphisms between X and Y, by the definition a function $f : X \to Y$ is a homeomorphisms if and only if .

(i) f is a objective.

(ii) f is continuous

(iii) f^{-1} is also continuous.

Definition 2.8.4

A differentiable manifold of dimension N is a set M and a family of injective mapping $x_\alpha \subset R^n \to M$ of open sets $u_\alpha \in R^n$ into M such that.

(i) $u_\alpha x_\alpha(u_\alpha) = M$

(ii)For any α, β with $x_\alpha(u_\alpha) \cap x_\beta(u_\beta)$

(iii)the family (u_α, x_α) is maximal relative to conditions the pair (u_α, x_α) or the mapping x_α with $p \in x_\alpha(u_\alpha)$ is called a parameterization , or system of coordinates of M , $u_\alpha x_\alpha(u_\alpha) = M$ the coordinate charts (U, φ) where U are coordinate neighborhoods or charts, and φ are coordinate homeomorphisms transitions are between different choices of coordinates are called transitions maps.

$\varphi_{i,j} : (\varphi_j \circ \varphi_i^{-1})$

Which are anise homeomorphisms by definition, we usually write $x = \varphi(p), \varphi : U \to V \subset R^n$ collection U and $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ for coordinate charts with is $M = \cup U_i$ called an atlas for M of topological manifolds. A topological manifold M for which the transition maps $\varphi_{i,j} = (\varphi_j \circ \varphi_i)$ for all pairs φ_i, φ_j in the atlas are homeomorphisms is called a differentiable, or smooth manifold, the transition maps are mapping between open subset of R^m, homeomorphisms between open subsets of R^m are C^∞ maps whose inverses are also C^∞ maps , for two charts U_i and U_j the transitions mapping.

$\varphi_{i,j} = (\varphi_j \circ \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$

And as such are homeomorphisms between these open of R^m, for example the differentiability $(\varphi'' \circ \varphi^{-1})$ is achieved the mapping $(\varphi'' \circ (\tilde{\varphi})^{-1})$ and $(\tilde{\varphi} \circ \varphi^{-1})$ which are homeomorphisms since $(A \approx A'')$ by assumption this establishes the equivalence $(A \approx A'')$, for example let N and M be smooth manifolds n and m respecpectively, and let $f : N \to M$ be smooth mapping in local coordinates $f' = (\psi \circ f \circ \varphi^{-1}) : \varphi(U) \to \psi(V)$,with respects charts (U, φ) and (V, ψ), the rank of f at $p \in N$ is defined as the rank of f' at $\varphi(p)$ i.e. $rk(f)_p = rk(J\, f')_{\varphi(p)}$ is the

Definition 2.8.5

Let I_n be the identity map on R^n, then $\{R^n, I_n\}$ is an atlas for R^n indeed, if U is any nonempty open subset of R^n, then $\{U, I_n\}$ is an atlas for U so every open subset of R^n is naturally a C^∞ manifold.

Example 2.8.6

The n-space is a manifold of dimension n when equipped with the atlas $A_1 = \{(U_i, \varphi_i), (V_i, \psi_i), |1 \le i \le n+1| \}$ where for each $1 \le i \le n+1$.

$$(17) \quad \begin{cases} U_i = \{(x_1, \ldots, x_{n+1}) \in S^n, x_1 \ge 0\} \ \varphi_i(x_1, \ldots, x_{n+1}) \\ = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \end{cases}$$

OERATOR GEOMETRIC ON RIEMANNIAN MANIFOLDS

3.1 Vector Analysis one Method Lengths]

Classical vector analysis describes one method of measuring lengths of smooth paths in R^3 if $v:[0,1]\to R^3$ is such a paths, then its length is given by length $v=v(t)\,dt$. Where $[v]$ is the Euclidean length of the tangent vector (t), we want to do the same thing on an abstract manifold, and we are clearly faced with one problem, how do we make sense of the length $|v(t)|$ obviously , this problem can be solved if we assume that there is a procedure of measuring lengths of tangent vectors at any point on our manifold The simplest way to do achieve this is to assume that each tangent space is endowed with an inner product. (Which can vary point in a smooth).

Definition 3.1.1

A Riemannian manifold is a pair $(M.g)$ consisting of a smooth manifold M and a metric g on the tangent bundle, i.e a smooth symmetric positive definite tensor field on M. The tensor g is called a Riemannian metric on M. Two Riemannian manifold are said to be isometric if there exists a diffeomorphism $\phi:M_1\to M_2$ such that $\phi^*:g_1=g_2$ If $(M.g)$ is a Riemannian manifold then, for any $x\in M$ the restriction $g_x:T_x(M_1)\times T_x(M_2)\to R$. Is an inner product on the tangent space $T_x(M)$ we will frequently use the alternative notation $(\cdot,\cdot)_x=g_x(\cdot,\cdot)$ the length of a tangent vector $v\in T_x(M)$ is defined as usual $|v|_x=g_x(v,v)^{1/2}$. If $v:[a,b]\to M$ is a piecewise smooth path, then we defined is length by $L(v)=\int_a^b|v(t)|dt$. If we choose local coordinates $(x^1,....,x^n)$ on M, then we get a local description of g as.

(18) $[g=g_{ij}(dx^i,dx^j)],\left[g_{ij}=g\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\right]$

Proposition 3.1.2

Let be a smooth manifold, and denote by R_M the set of Riemannian metrics on M then R_M is a non –empty convex cone in the linear of symmetric tensor

Example 3.1.3

Let (M,g) be Riemann manifold and $S\subset M$ a sub manifold if $S\to M$, denotes the natural inclusion then we obtain by pull back a metric on $S, g^S=i\,g^*=(g/S)$. For example, any invertible symmetric $(n\times n)$ matrix defines a quadratic hyper surface in R^n by $H_A=\{x\in R^n,(A_x,x)=1\}$ where $[\cdot,\cdot]$ denotes the Euclidean inner on R^n, H_A has a natural.

Example 3.1.4

The Poincare model of the hyperbolic plane is the Riemannian manifold (D,g) where D is the unit open disk in the plan R^n and the metric g is given by.

(18) $g=\left[\frac{1}{1-x^2-y^2}\right](dx^2+dy^2)$

Example 3.1.6

Consider a lie group G, and denote by L_G its lie algebra then any inner product $\langle\cdot,\cdot\rangle$ on L_G induces a Riemannian metric $h=\langle\cdot,\cdot\rangle_g$ on G defined by.

(19) $\begin{cases} h_g(x,y)=\langle x,y\rangle_g=\langle L_g^{-1}\rangle * X,(L_g^{-1})*Y \\ \forall:g\in G, X\ \ y\in T_g(G)\end{cases}$

Where $(L_g^{-1})_*:T_g(G)\to T_1(G)$ is the differential at $g\in G$ of the left translation map L_g^{-1}. One checks easily that check easily that the correspondence $G\in g\to\langle\cdot,\cdot\rangle$ is a smooth tensor field, and it is left invariant (i,e) $L_g^* h=h\ \ \forall g\in G$. If G is also compact, we can use the averaging technician to produce metrics which are both left and right invariant.

3.2 The Levi-Cavite Connection

To continue our study of Riemannian manifolds we will try to follow a close parallel with classical Euclidean geometry the first question one may ask is whether there is a notion of "straight line" on a Riemannian manifold. In the Euclidean space R^3 there are at least ways to define a line segment a line segment is the shortest path connecting two given points a line segment is a smooth path $v:[0,1]\to R^3$ satisfying $\ddot{v}(t)=0$. Since we have not said anything about calculus of variations which deals precisely with problems of type.

(i) We will use the second interpretation as our starting point, we will soon see however that both points of view yield the same conclusion.

(ii) Let us first reformulate as know the tangent bundle of R^3 is equipped with a natural trivialization, and as such it has a natural trivial connection ∇^0 defined by. $\nabla_i^0(\partial_j)=0\ \forall: i,j$ Where,

(19) $$(\nabla^0 \partial_j = 0), \forall i,j \left(\partial_j = \frac{\partial}{\partial x_i}, \nabla_i = \nabla_{\partial_i} \right)$$

All the Christ off symbols vanish, moreover, if g_0 denotes the Euclidean metric, then.

(20) $$\begin{cases} (\nabla^0{}_i g_0)(\partial_j,\partial_k) = \nabla^0_{ijk} - g_0(\nabla_i^0\partial_j,\partial_k) - g_0(\partial_j,\nabla^0\partial_k) \\ \nabla^0_{V(t)}V(t)=0 \end{cases}$$

So that the problem of defining "lines" in a Riemannian manifold reduces to choosing a "natural" connection on the tangent bundle of course, we would like this connection to be compatible with the metric but even so, there infinitely many connections to choose from. The following fundamental result will solve this dilemma.

Proposition 3.2.1

Let (M,g) be a Riemannian manifold for any compact subset $\subset TM$ there exists $\varepsilon \geq 0$ such that for any $(x,X) \in k$ there exists a unique geodesic $V = V_X X : (-\varepsilon,\varepsilon) \to M$ such that $V(0)=x, V(0)=X$

One can think of a geodesic as defining a path in the tangent bundle $t \to (V(t), V(t))$. The above proposition shows that the geodesics define a local flow ϕ on $T(M)$ by

(21) $$\phi^t(x.X) = (V(t), V(t))\ , V_X X$$

Definition 3.2.2

The local flow defined above is called the geodesic flow the Riemannian manifold (M,g) when the geodesic low is global flow i,e any $V_X X$ is defined at each moment of t for any $(x,X) \in T(M)$, then the Riemannian manifold is call geodetically complete .

Definition 3.2.3

Let L be finite dimensional real lie algebra, the killing paring or form is the bilinear map.

(22) $$\left\{ \begin{cases} K: L \times L \to R, K(X,Y) = -tr\,(ad(X).ad(Y)) \\ \forall: X,Y \in L \end{cases} \right\}$$

The lie algebra L is said to be semi simple if killing paring is a duality, a lie group G is called semi simple if its lie algebra is semi simple.

Proposition 3.2.4

Let (M,g) and $q \in M$ as above .Then there exists $r \geq 0$ such that the exponential map.($\exp_q : D_q(r) \to M$ Is a diffeomorphism on to. The supermom of all radii r with this property is denoted $P_M(q)$.

Definition 3.2.5

The positive real number $P_M(q)$ is called the infectivity radius of M at q the infemur. $P_M = \inf_q [P_M(q)]$

Is called the infectivity radius of M

Lemma 3.2.6

The Freshet differential at $0 \in T_q(M)$ of the exponential map, $D_0 \exp_q : T_q(M) \to T \exp_q(0) M = T_q(M)$. Is the identity $T_q(M) \to T_q(M)$

Theorem 3.2.7

Let q,r and ε as in the previous and consider the unique geodesic $r:[0,1] \to M$ of length $\leq \varepsilon$, joining two points $B_r(q)$ .if $w:[0,1] \to M$ is a piecewise smooth path with the same endpoint as then.

(23) $$\int_0^1 |(t)|\, dt \leq \int_0^1 |\, w(t)\,|\, dt$$

With equality if and only if $w([0,1]) = ([0,1]$ Thus ɤ is the shortest path, joining its endpoints.

3.4: Riemannian Geometry

Definition 3.4.1 Riemannian Metrics

Differential forms and the exterior derivative provide one piece of analysis on manifolds which, as we have seen, links in with global topological questions. There is much more on can do when on introduces a Riemannian metric. Since the whole subject of Riemannian geometry is a huge to the use of differential forms. The study of harmonic from and of geodesics in particular, we ignore completely hare questions related to curvature.

Definition 3.4.2 Metric Tensor

In informal terms a Riemannian metric on a manifold M is a smooth varying positive definite inner product on tangent space T_x. To make global sense of this note that an inner product is a bilinear form so at each point x, we want a vector in tensor product. $T_x^* \otimes T_x^*$ We can put, just as we did for exterior forms a vector bundle striation on. $T^*M \otimes T^*M = \bigcup_{x \in M} T_x^* \otimes T_x^*$. The conditions we need to satisfy for a vector bundle are provided two facts we used for the bundle of p-forms each coordinate system $(x_1, \ldots\ldots, x_n)$ defiance a basis $dx_1, \ldots\ldots, dx_n$ for each T_x^* in the coordinate neighborhood and the n^2 element. $dx_i \otimes dx_j \quad 1 \le i, j \le n$. Given a corresponding basis for $T_x^* \otimes T_x^*$. The Jacobean of a change of coordinates defines an invertible linear transformation. $J : T_x^* \to T_x^*$ And we have a corresponding.

(24) $\quad J \otimes J = (T_x^* \otimes T_x^*) \to (T_x'^* \otimes T_x'^*)$

Definition 3.4.3 Local Coordinate System A Riemannian metric on manifold M is a section g of $T_x^* \otimes T_x^*$ which at each point is symmetric and positive definite. In a local coordinate system we can write.

(25) $$g = \sum_{i,j} g_{ij}(x)\, dx_i dx_j$$

Where $g_{i,j}(x) = g_{j,i}(x)$ and is a smooth function, with $g_{i,j}(x)$ positive definite. Often the tensor product symbol is omitted and one simply writes. $g = \sum_{i,j} g_{ij}(x) dx_i dx_j$

Definition 3.4.4

A diffehomorphism $F : M \to N$, between two Riemannian manifolds is an isometric if $F^* g_N = g_N$

Definition 3.4.5

Let M a Riemannian manifold and $\gamma : [0,1] \to M$ a smooth map i,e a smooth curve in M. The length of curve is $L(\gamma) = \int_0^1 \sqrt{g\,(\gamma, \gamma)}\; dt$. Where $\gamma'(t) = D_{\gamma_t}\left(\frac{d}{dt}\right)$, with this definition, any Riemannian manifold is metric space define.

(26) $\quad d(x, y) = \inf \{ L(\gamma) \in R : \gamma(t) = y \}$

are Riemannian an manifold space.

Proposition 3.4.6

Consider any manifold M and its cotangent bundle $T^*(M)$, with projection to the base $p : T^*(M) \to M$, let X be tangent vector to $T^*(M)$ at the point $\zeta \in T_a^* M$ then $D_p(X) \in T^*(M)$ so that $\varphi(X) = \zeta_a(D_p(x))$ defines a conical a conical 1-form φ on $T^*(M)$ in coordinates $(x, y) \to \sum_i y_i\, dy$ the projection p is $p(x, y) = x$ so if $X = \left(\sum a_i \frac{\partial}{\partial x_i} \right) + \left(\sum b_i \frac{\partial}{\partial y_i} \right)$ so if given take the exterior derivative $w = -d\varphi = \sum dx_i \wedge dy_i$ which is the canonical 2-from on the cotangent bundle it is non-degenerate, so that the map $X \to (i \times w)$ from the tangent bundle of $T^*(M)$ to its contingent bundle is isomorphism. Now suppose f is smooth function and $T^*(M)$ its derivative is a 1-form do. Because of the isomorphism a above there is a unique vector field X on $T^*(M)$ such that $df = (i \times w)$ from the g another function with vector field Y,

Definition 3.4.7

The vector field X on $T^*(M)$ given by $I_i w = dH$ is called the geodesist flow of the metric g.

Definition 3.4.11

If $\gamma : (a,b) \to T^*(M)$ Is an integral curve of the geodesic flow. Then the curve $P(\gamma)$ in (M) is called ageodesic. In locally coordinates, if the geodesic flow.

(27) $$X = \left(a_i \frac{\partial}{\partial x_i} \right) + \left(b_j \frac{\partial}{\partial y_j} \right)$$

$T_p M$ at every point p of M, then $T_p M$ is isomorphic to $M \times R^n$ m here isomorphic means that TM and $M \times R^n$ are homeomorphism as smooth manifolds and for every $p \in M$, the homeomorphism restricts to between the tangent space $T_p M$ and vector space $\{P_i\} \times R^n$.

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The basic notions on analytic geometry knowledge of calculus, including the geometric formulation of the notion of the differential and the inverse function theorem. The differential Geometry of surfaces with the basic definition of differentiable manifolds, starting with properties of covering spaces and of the fundamental group and its relation to covering spaces.

REFERENCES

1. Osman. MohamedM, Basic integration on smooth manifolds and application maps with stokes theorem, http//www.ijsrp.org-6-januarly2016.
2. sman.Mohamed M, fundamental metric tensor fields on Riemannian geometry with application to tangent and cotangent, http//www.ijsrp.org-6januarly2016.
3. Osman. Mohamed M, operate theory Riemannian differentiable manifolds, http//www.ijsrp.org- 6januarly2016.
4. J.Glover, Z, pop-stojanovic, M.Rao, H.sikic, R.song and Z.vondracek, Harmonic functions of subordinate killed Brownian motion, Journal of functional analysis 215(2004)399-426.
5. M.Dimitri, P.Patrizia, Maximum principles for inhomogeneous Ellipti inequalities on complete Riemannian Manifolds, Univ. deglistudi di Perugia, Viavanvitelli 1,06129 perugia, Italy e-mails: Mugnai@unipg,it , 24July2008.
6. Noel.J.Hicks. Differential Geometry, Van Nostrand Reinhold Company450 west by Van N.Y10001.
7. L. Jin, L. Zhiqin, Bounds of Eigenvalues on Riemannian Manifolds, Higher eduationpress and international press Beijing-Boston, ALM10, pp. 241-264.
8. S.Robert. Strichaiz, Analysis of the laplacian on the complete Riemannian manifolds. Journal of functional analysis52, 48, 79(1983).
9. P.Hariujulehto. P.Hasto, V.Latvala, O.Toivanen, The strong minmum principle for quasisuperminimizers of non-standard growth, preprint submitted to Elsevier, june 16, 2011.-Gomez, F, Rniz del potal 2004.
10. Cristian, D.Paual, The Classial Maximum principles some of ITS Extensions and Applicationd, Series on Math. And it Applications, Number 2-2011.
11. H.Amann, Maximum principles and principlal Eigenvalues, J.Ferrera. J. Lopez
12. J.Ansgarstaudingerweg 9.55099 mainz, Germany e-mail: juengel@mathematik.uni-mainz. De, U.Andreas institute fiir mathematic, MA6-3, TU Berlin strabe des 17.juni. 136, 10623 Berlin, Germany, e-mail: unterreiter@math.tu-berlin.
13. R.J. Duffin, The maximum principle and Biharmoni functions, journal of math. Analysis and applications 3.399-405(1961).

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