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Original Research Article

Fixed Points of Quasi Contractive Mapping using Ishikawa Iteration

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ABSTRACT

A map has a fixed point at P. If fixed point theorems have useful applications in analysis. Some of the iterative methods which have been studied are related to S. Banach, W.R. Mann, J. Riemermann, W.G. Dastonand a host of other mathematics.
Studies by Prof. S. Ishikawa and Prof. B.E. Rhoads, throw new light on the iteration process of W.R. Mann, Prof. Ishikawa studied by the following iteration process.
For a subset E of an Ailbert space H, if and only if the sequence generated by $x_{n-1} = (1-c_n)x_n + c_nT_{x_n}, n \geq 1$. Where ($c_n$) are real sequence in [0, 1].

Key Words: Fixed point, metric space, picard iteration , Ishikawa iteration.

1.1. INTRODUCTION

Prof B.E. Khoades has shown in [17] that from amongst various generalizations of Banach's contraction principle the definition of quasi contraction by circle [1] is one of the most general contractive definition for which Picard's Iteration gives a unique fixed point. We recall the definition of a Quasi contractive method which states that if there exists a constant k, $0 \leq k < 1$ such that for each x, $y \in E$. $\| T_X - T_y \| \leq k \max \{\| x - y \|, \| x - T_X\|, \|y - T_y\|, \| y - T_X\|\}$......(x). In [3] Hu." has shown that most of the results of [17] which use M (x_{11}, C_n, T) can be extended to Ishikawa's iteration scheme I (X_1, C_n, d_n, T). Thus I (X_1, C_n, d_n, T) becomes a larger class of fixed point iteration method. However he [4 theorem 9] posed an open question whether mann iterative process M (X_1, C_n, T) can be replaced by that of Ishikawa I (X_1, C_n, d_n, T) for quasi contractive mapping in this chapter our purpose is to show that in a Hilbert space I (X_1, C_n, d_n, T) converges to the fixed point of a Quasi contractive map. This is embodied is theorem I below. Theorem 2 provides a generalization of theorem.

1.2 MAIN RESULTS

Theorem: Let E be a compact and convex subset of a Hilbert space. H and T be a quesi contractive self map on E. Let a sequence (X_n) be defined iteratively on E by

$X_1 \in E$, $X_{n+1} = (1 - C_n) X_n + C_nT [(1-d_n) X_n + d_nTX_n]$......... (1)

Where (C_n) and (T_n) are sequences of real numbers such that

(i) $0 \leq C_n \leq d_n \leq 1$

(ii) $\lim_{n \to \infty} d_n = 0$

(iii) $\sum_{n=1}^{\infty} C_n d_n = \infty$

Then (X_n) converges to the fixed point of T.

Pf : Since T is quasi contractive by circle [1] it has a unique fixed point P say. Hence from (*) putting P for y we have for each $x \in E$,

$\| T_x - P \| \le k \max \{\| x - p \|, \|x - T_x\|\}$ (2)

writing $Y_n = (1-d_n)X_n + d_n$ (3)

we can express X_{n+1} in (1) as

$X_{n+1} = (1 - C_n) X_n + C_n T_{yn}$ (4)

We know [33] that for any X, Y, Z in a Hilbert space and for any real number t.

$\| t_x + (1-t)y-z \|^2 = t \| x - z \|^2 + (1-t) \| y - z \|^2 - t (1 - t) \| x - y \|^2$ (5)

Hence from (3) and (4) we have the following relation.

$\|x_{n+1}-p\|^2 = 1 - C_n \| X_n-p\|^2 + C_n \|T_{yn}-p\|^2 - C_n (1-C_n) \|X_n-T_{yn}\|^2$ (6)

$\|Y_n-T_{yn}\|^2 = (1-d_n) \|X_n-T_{yn}\|^2 + d_n \| T_{xn}-T_{yn} \|^2 - d_n (1-d_n) \|X_n - T_{xn}\|^2$... (7)

$\|Y_n-p\|^2 = (1-d_n) \|x_n - p\|^2 + d_n \|T_{xn}-p\|^2 - dn (1-d_n) \|x_n-T_{xn}\|^2$..... (8)

Also by (2)

$\|T_{yn}-p\| \le k \max \{\|y_n-p\|, \|y_n-T_{yn}\|\}$

and

$\|T_{xn}-p\| \le k \max \{\|x_n - p \|, \|x_n -T_{xn} \|\}$

Let $S_1 = \{n \in N: \|T_{yn} - p \| \le k \| Y_n-p \|\}$

and $S_2 = \{n \in N: \|T_{yn} - p \| \le k \| Y_n-T_{yn} \|\}$

where N denote the set of positive integers. Obviously $S_1 \cup S_2 = N$

Suppose $n \in S_1$. Then using (8) we have

$\|T_{yn}-p\|^2 \le k^2 \|Y_n - p\|^2$

$=k^2(1-d_n) \|x_n - p\|^2 + k^2 dn \|T_{xn}-p\|^2 - k^2 dn (1-dn) \|Xn-T_{xn}\|^2$ ...... (10*)

If in (10) $\|T_{xn}-p\| \le k \|x_n-p\|$ holds. Then form 10*.

$\|T_{yn} - p\|^2 \le [k^2(1-d_n) + K^4 d_n] \|X_n - p\|^2 - k^2 d_n(1-d_n) - \|x_n-T_{xn}\|^2$

$\le \|x_n-p\|^2 - k^2 d_n(1-d_n) \|x_n-T_{xn}\|^2$

Thus for all $n \in S_1$

$k^2 c_n d_n(1-d_n-k^2) \|x_n - T_{xn}\|^2 \|x_n-p\|^2 - \|x_{n+1} - p\|^2$...... (11)

Now supposing that $n \in S_2$ we have by (7)

$\| T_{yn}- p\|^2 \le k^2 \|y_n-T_{yn}\|^2$

$= k^2 (1-d_n) \| X_n -T_{yn}\|^2 + k^2 d_n \|T_{xn} -T_{yn}\|^2 - k^2 d_n(1-d_n) \|x_n -T_{xn} \|^2$..... (12)

Since $\|x_n-T_{yn}\| = dn \|x_n-T_{xn}\|$

$\|y_n - T_{yn}\| = (1-d_n) \|x_n-T_{xn} \|$ and

T satisfies (*) we have

$\|T_{xn} -T_{yn}\| \le k\max \{\|x_n - T_{xn}\|, \| y_n-T_{yn}\|, \|x_n-T_{xn}\|\} = k A_n$

Where A_n denotes the maximum of the set.

Let $S_2^1 = \{n \in S_2 : A_n = \|x_n-T_{xn}\|\}$

$S_2^{11} = \{n \in S_2 : A_n = \|y_n-T_{yn}\|\}$

$S_2^{111} = \{n \in S_2 : A_n = \|x_n-T_{yn}\|\}$

Clearly $S_2 = S_2' \cup S_2'' \cup S_2'''$

Now if $n \in S_2'$ i.e. if

$\|T_{xn}-T_{yn}\| \le k \|x_n-T_{xn}\|$, then form (12)

$\|T_{yn}-p\|^2 \le k^2(1-d_n) \|x_n-T_{yn}\|^2 - x^2 d_n(1-d_n-k^2) \|x_n-T_{xn}\|^2$

and using (6) we get

$\| x_{n+1} - p \|^2 \le (1-c_n) \|x_n - p\|^2 + c_n k^2 (1-d_n) \|x_n-T_{yn}\|^2 - c_n k^2 d_n (1-d_n-k^2) \|x_n-T_{xn}\|^2 c_n(1-c_n) \|x_n-T_{yn}\|^2$

$= (1-c_n) \|x_n - p\|^2 - k^2 c_n d_n (1-d_n-k^2) \|x_n-T_{xn}\|^2 - c_n [1-c_n-k^2 (1-d_n)] \| x_n-T_{yn}\|^2$

$\le \|x_n - p\|^2 - k^2 c_n d_n (1-d_n-k^2) \|x_n - T_{xn}\|^2$

Since $k^2 (1-d_n) < (1-d_n) \le (1-c_n)$

or $k^2 c_n d_n (1-d_n-k^2) \|x_n -T_{xn} \|^2 \le \| x_n - p\|^2 - \|x_{n+1} - p\|^2$ (13)

I $n \in S_2''$ i.e. if $\| T_{xn} -T_{yn} \| \le k \|y_n-T_{yn}\|$

Thenusing (7) we obtain

$\| T_{xn} -T_{yn} \|^2 \le k^2 \|y_n-T_{yn}\|^2$

$\le k^2(1-d_n) \|x_n-T_{yn}\|^2 + k^2 d_n \|T_{xn} -T_{yn}\|^2$

$- k^2 d_n(1-d_n) \|x_n-T_{xn}\|^2$

and hence

$\|T_{xn} -T_{yn}\|^2 \le [k^2(1-d_n) / (1-k^2 d_n)] \|x_n-T_{yn}\|^2$

$-[k^2 d_n(1-d_n) / (1-k^2 d_n)] \|x_n-T_{yn}\|^2$

Then by (12)

$\|T_{yn} - p\|2 \leq k^2(1-d_n)\ [(1+k^2 d_n\ /\ (1-k^2 d_n)]\ \|x_n - T_{yn}\|^2$

$- k^2 d_n(1-d_n)\ [1+k^2 d_n\ (1-k^2 d_n)]\ \|x_n - T_{xn}\|^2$

$= [k^2(1-d_n) / (1-k^2 d_n)]\ \|x_n - T_{yn}\|^2$

$- [k^2 d_n(1-d_n) / (1-k^2 d_n)]\ \|x_n - T_{xn}\|^2$

From which we obtain on using (6)

$\|x_{n+1} - p\|^2 \leq (1-c_n)\ \|x_n - p\|^2 + C_n\ [k^2\ (1-d_n) / (1-k^2 d_n)]$

$\|x_n - T_{yn}\|^2 - c_n[k^2 d_n\ (1-d_n) / (1-k^2 d_n)]$

$\|x_n - T_{xn}\|^2 - c_n\ (1-c_n)\ \|x_n - T_{yn}\|^2$

$= (1-c_n)\ \|x_n - p\|^2 - [k^2 c_n d_n\ (1-d_n)\ /\ (1-k^2 d_n)]\ \|x_n - T_{xn}\|^2$

$- [c_n\ (1 - k^2 - c_n\ (1-k^2 d_n))\ /\ (1-k^2 d_n\ ]\ \|x_n - T_{yn}\|^2$

Since $C_n \rightarrow 0$ as $n \rightarrow \infty$ there exists an $n_0 \in$ N such that $1- k^2 > c_n$ for all $n \geq n_0$. Thus $n \geq n_0$ the last form on the right hand sides of the above expression is positive and hence we get $n > n_0$.

$[k^2 c_n d_n\ (1-d_n)\ /\ (1-k^2 d_n)]\ \|x_n - T_{xn}\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2$...... (14)

If $n \in S_2''$ i.e. iff

$\|T_{xn} - T_{yn}\| \leq k\ \|x_n - T_{yn}\|$, then form (6) and 12 we obtain.

$\|x_{n+1} - p\|^2 \leq (1-c_n)\ \|x_n - p\|^2 + C_n\ [k^2\ (1-d_n)\ \|x_n - T_{yn}\|^2$

$+ k^2 d_n k^2 \|x_n - T_{yn}\|^2 - k^2 d_n\ (1-d_n)\ \|x_n - T_{xn}\|^2]$

$- c_n(1-c_n)\ \|x_n - T_{yn}\|^2$

Hence for $n \geq n_0$ we get as before

$k^2 c_n d_n\ (1-d_n)\ \|x_n - T_{xn}\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2$

Since $1-d_n-k^2 \leq 1-d_n \leq (1-d_n)\ /\ (1-k^2 d_n)$ for all n.

We obtain from the inequalities (13) (14) and (15) that for all $n \in S_2$ and $n \geq n_0$

$k^2 c_n d_n\ (1-d_n-k^2)\ \|x_n - T_{xn}\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2$

Since this inequality holds for all $n \in S_1$ sec (11) it follows that (11) holds for all $n \in N$, $n \geq n_0$

Now choosing $m \geq n_0$ and adding the in equalities (11) for values m, m + 1.....n of n.

We obtain

$\sum_{j=m}^{n} l^2 cj\ dj(1-dj-k)^2\ \|x_j - T_{xj}\|^2 \leq \|x_m - p\|^2 - \|x_{n+1} - p\|^2$ ..... (16)

Since $dj \rightarrow 0$ as $J \rightarrow \infty$, $k^2\ (1 - dj - k^2)$ is +ve and bounded away from zero. The fact that right hand side of the above inequality is bounded and that $\sum_{j-1}^{\infty} cjdj = \infty$ imply that $\lim_{n\to\infty} iff\ \|X_n - T_{xn}\| = 0$. Hence by compactness of E. It follows that there exists a subsequence (X_{nk}) such that and $\lim_{n\to\infty} X_{nk} = q$ and $\|q - T_q\| = 0$ i.e. $T_q = q$. But by uniqueness of a fixed point of T. q = p. Again since the sequence $(\|x_n - p\|)$ is monotonically decreasing (as evident from (11)) and $\lim_{n\to\infty} X_{nk} = p$ it ultimately follows that $\lim_{n\to\infty} X_n = p$.

This completes the proof.

1.3 FURTHER GENERALISATION

B. Fisher [7p, 8p] established the existence of a common fixed point of a pair of commuting mapping S and T satisfying the inequality.

$\|S_x - T_y\| \leq k \max \{\|x - y\|, \|x - T_y\|\ \|y - S_x\|\ \|x - S_x\|\ \|y - T_y\|\}$

She proved the following theorem.

Theorem - 1

Let S and T be commuting mapping of a complete metric space (x, d) into itself satisfying (2) (with d (x, y) = $\|x - y\|$ of cause) for all X, Y in X, where $0 \leq k < 1$

and the inequality. Sup {d (S^{r+1} T^nx, S^rT^nx), d (S^r T^{n+1} X, S^rT^nx): r, n = 0, 1, 2.....} < For some particular X in X. Then S and T have a unique common fixed point Z. Further Z is the unique fixed point of S and T.

We know that from (16) an iteration involving two mapping S and T which satisfy (17) converges to their common fixed point. This result reduces to Theorem-1 in the case S and T.

Theorem - 2

Let E be a copact and convex subset of a Hilbert space H. Let S and T be a pair of commuting self mapping S on E satisfying the inequality (9) for all x, y $\in$ E and $0 \leq k < 1$. Let the sequence (Xn) be defined on E by the iteration $X_1 \in E$, $X_{n+1} = (1-c_n) X_n + C_nT [(1-d_n) X_n + d_nS_{xn}]$, $n \geq 1$........ (18). Where (Cn) and (dn) are real sequences satisfying conditions (1), (11) of theorem 1. Then (Xn) convergence to the common fixed point of S and T.

Proof

Since E is compact and S and T are commuting mappings satisfying (17), the conditions of Theorem A of Fisher are satisfied where there exists a unique common fixed point P. Say of S and T. The proof of convergence of (Xn) to P is similar to the proof of Theorem 1 and hence omitted.

1.4 CONCLUSION

Iteration (18) of theorem 2 above shows that Ishikawa iteration scheme I (X_1,C_n,D_n,T) can be generalized by introducing more number of mapping which can be used to yield common fixed point (s) of the mappings Investigation in this direction has been carried out in Chapter III and Chapter IV.

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